MODULE 2

continuous Random variable

A random variable X is said to be continuous if it's range is uncountable ie, the random variable X takes the values on a interval is said to be continuous.

* probability distribution of a discrete random variable X is completely determined if we specify $P(X = x_i)$ for each possible value x_i as X (since R_x (Range space) for a discrete r.v is countable set.
* But ; in the case of continuous r.v, the above definition of probability distribution is not possible, Here arise the term probability density function (pdf).

Probability Density function (pdf)

If X is a continuous r.v such that,

$P\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx\} = f(x)\,dx$

then $f(x)$ is called the probability density function of X, provided $f(x)$ satisfies the following conditions,

(1) $f(x) \geq 0$ for all $x \in R_x$ & (nonnegativity of pdf)

(2) $\int_{R_x} f(x)dx = 1$ (Normalization condition of pdf)

* In connection with continous r.v probabilities are given by integral evaluated over the ~~catogra~~ intervels,

where as $P(X=C) = 0$ for any real number

Bcoz of this property, doesnot matter whether be included the end points of the interval from a to b.

ie,

* If X is a continous r.v and a & b are real constants with $a<b$ then,

$$P(a<x<b) = P(a \leq x \leq b) = P(a<x \leq b) = P(a \leq x \leq b)$$
$$= \int_a^b f(x)dx \text{ (Area under the graph of } y=f(x)\text{)}$$

* The curve $y=f(x)$ is called the probability curve of the R.V X.

Example.

The noon time temperature in a city on any day during summer is b/w 30 and 35 degrees. Let T denote the temperature on a day, we may assume that T is a r.v with possible values given by the interval [30, 35].

what is the probability that T = 33? Any interval containing 33 such as (32.999, 33.001) however small it may be still contains uncountably infinite possible values of T, and it is near to impossible that the temperature will hit exactly 33 with infinite decimal precision. so the relative frequency of the event T = 33 should be takes as zero.

ie, $P(T = 33) = 0$.

Properties of pdf

Let X be a continous r.v with pdf $f(x)$, then

(1) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

ie, Total area under the curve $y = f(x)$ is 1.

(2) $f(x) \geq 0 \;\; \forall x$

Thus the graph of $y = f(x)$ will never go below the x axis.

Note:

If f is the PMF of a discrete r.v X, then $f(x_0)$ gives the probability of $X = x_0$. on the other hand, if f is ~~not~~ PDF of a continuous r.v, $f(x_0)$ is not a probability but the probability density of X at x_0. we get probability when $f(x)$ is integrated.

Cumulative distribution function (cdf)

If X is a continuous random variable then the fn given by,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt \quad , -\infty < x < \infty$$

where $f(t)$ is the value of the random variable at $x = t$ is called the cdf of the r.v X.

properties.

1. $F(x)$ is a nondecreasing fn of x

   ie, if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$

2. $F(-\infty) = 0$ & $F(\infty) = 1$

3. $f(x) = \frac{d}{dx}(F(x))$, ie, $f(x) = F'(x)$,

here, $f(x)$ & $F(x)$ are respectively represents the pdf and cdf of the r.v X.

4. If X is a continuous r.v

(1) $p(a<x<b) = \int_a^b f(x)\,dx = [F(x)]_a^b$

$= F(b) - F(a)$

$F'(x) = f(x)$

$\int f(x)\,dx = F(x)$

(2) $p(X \leq a) = F(a)$

(3) $p(X \geq b) = 1 - p(X<b)$

$= 1 - p(X \leq b)$

$= 1 - F(b)$

2. If $f(x) = \begin{cases} xe^{-x^2/2}, & x \geq 0 \\ 0, & x<0 \end{cases}$

S.T $f(x)$ is a pdf of a continuous r.v.

A. We've to s.t

$f(x) \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here,

when $x \geq 0$, $xe^{-x^2/2} \geq 0$, cause exponential d a nu is always positive. & $\therefore$ product of 2 positive numbers is also a +ve number.

$\therefore$ $f(x) \geq 0$ $\forall x$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} x e^{-x^2/2}\,dx$$

$$= 0 + \int_{0}^{\infty} x e^{-x^2/2}\,dx$$

$t = x^2/2$

$dt = x\,dx$

$x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$$= \int_{0}^{\infty} e^{-t}\,dt$$

$$= -\left[e^{-t}\right]_{0}^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -[0 - 1]$$

$$= 1$$

clearly,

$\therefore$ $f(x)$ is a pdf of contincos r.v X.

Mean and Variance of a continous R.V

* Let X be a continuous R.V with pdf $f(x)$. Then the mean of X also Known as $E(X)$ is defined as,

$$\boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\,dx}$$

* mean of X or $E(X)$ is also denoted by 'μ'.

* $$\boxed{E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx}$$

$$Var(X) = E[(X-\mu)^2] = E(X^2) - (E(X))^2$$

Q. A continous R.V X has pdf $f(x) = kx^2 e^{-x},\ x \geq 0$

Find k, mean & variance.

A). By the normalisation property of pdf,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

here $0 \leq x \leq \infty$

$$\therefore \int_{0}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx^2 e^{-x}\,dx = 1$$

$$k\int_{0}^{\infty} x^2 e^{-x}\,dx = 1$$

$$k\left[x^2 - e^{-x} - 2xe^{-x} + 2 - e^{-x}\right]_0^{\infty} = 1$$

$$k\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k\left[-2e^{-\infty} - [-2e^{0}]\right] = 1$$

$$k[0+2] = 1$$

$$k = 1/2$$

$$\therefore f(x) = 1/2\, x^2 e^{-x},\ x \geq 0$$

Mean, $E(x) = \int_{0}^{\infty} x f(x)\,dx$

$$= \int_0^\infty x \cdot \frac{1}{2} x^2 e^{-x} \, dx$$

$$= \frac{1}{2} \int_0^\infty x^3 e^{-x} dx$$

$$= \frac{1}{2} \left[x^3 \cdot -e^{-x} - 3x^2 e^{-x} + 6x \cdot -e^{-x} - 6e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left[-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left[-6e^{-\infty} - [-6 \times 1] \right]$$

$$= 1/2 \; [0 + 6]$$

$$= 3$$

$E(x) = 3$

$$\text{var}(x) = \int_{-\infty}^{\infty} E(x^2) - (E(x))^2$$

$$E(x^2) = \int_0^\infty x^2 \times 1/2 \, x^2 e^{-x} \, dx$$

$$= \frac{1}{2} \int_0^\infty x^4 e^{-x} dx$$

$$= 1/2 \left[x^4 \cdot -e^{-x} - 4x^3 e^{-x} + 12x^2 \cdot -e^{-x} - 24x e^{-x} + 24 \cdot -e^{-x} \right]_0^\infty$$

$$= 1/2 \left[-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24x e^{-x} - 24 e^{-x} \right]_0^\infty$$

$$= 1/2 \left[0 - [-24 \times 1] \right]$$

$$= 12$$

$E(X^2) = 12$

$var(X) = 12 - 9$

$= \underline{\underline{3}}$

2. Find the mean & variance of a random variable x with following density function

$$F(x) = \begin{cases} K(10-x)x^2, & 0 \le x \le 1 \\ 0, & o.w \end{cases}$$

where K is a constant.

A). First we've to find the value of K, using normalisation property of pdf, ie,

$\int_{-\infty}^{\infty} f(x)\,dx = 1,$

here $0 \le x \le 1$

$\therefore \int_0^1 f(x)\,dx = 1$

$\int_0^1 K(10-x)x^2\,dx = 1$

$K \int_0^1 10x^2 - x^3\,dx = 1$

$K\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1$

$K\left[\frac{10}{3} - \frac{1}{4}\right] = 1$

$$K\left[37/12\right] = 1$$

$$K = 12/37$$

$$\therefore f(x) = \frac{12}{37}(10 - x)x^2$$

$$f(x) = \frac{12}{37}(10x^2 - x^3)$$

$$\text{mean} = E(X) = \int_0^1 x f(x)\, dx$$

$$= \int_0^1 x \cdot \frac{12}{37}(10x^2 - x^3)\, dx$$

$$= \int_0^1 \frac{12}{37}(10x^3 - x^4)\, dx$$

$$= \frac{12}{37}\int_0^1 10x^3 - x^4\, dx$$

$$= \frac{12}{37}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{12}{37}\left[\frac{10}{4} - 1/5\right]$$

$$= \underline{\underline{0.746}}$$

$$E(X) = 0.746$$

$$E(X^2) = \int_0^1 x^2 f(x)\, dx$$

$$= \int_0^1 x^2 \times \frac{12}{37}(10x^2 - x^3)\, dx$$

$= \int_0^1 \frac{12}{37} (10x^4 - x^5)\, dx$

$= \frac{12}{37} \int_0^1 10x^4 - x^5\, dx$

$= \frac{12}{37} \left[10\frac{x^5}{5} - \frac{x^6}{6} \right]_0^1$

$= \frac{12}{37} \left[2 - \frac{1}{6} \right]$

$= \underline{\underline{0.595}}$

$V(X) = E(X^2) - (E(X))^2$

$= 0.595 - (0.746)^2$

$= \underline{\underline{0.038}}$

2. Find the value of 'b' so that the following function is a valid pdf.

$$f(x) = \begin{cases} 2x, & 0 \le x \le b \\ 0, & o.w \end{cases}$$

Also find $P(X \geq 0.5)$ & $P(0 \le X < 4)$.

A). For a valid pdf, the given $f(x)$ must satisfy the normalization property,

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

here, $\int_0^b 2x\, dx = 1$

$x^2\Big]_0^b = 1$

$b^2 = 1$

$b = \pm 1$ ($\because$ probability is always ≥ 0)

$b = 1$ ✓

$\therefore f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & o.w \end{cases}$

$P(X \geq 0.5) = \int_{0.5}^{1} f(x)\,dx$

$= \int_{0.5}^{1} 2x\,dx$

$= x^2\Big]_{0.5}^{1}$

$= 1 - 0.25 = \underline{\underline{0.75}}$

$P(0 \leq X \leq 4) = \int_{0}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$

normalization property

$= 1 + 0$

$= 1$//

2. A continous R.V has a pdf $f(x) = 3x^2$, $0 \leq x \leq 1$

Find a & b such that.

(1) $P(X \leq a) = P(X > a)$

(2) $P(X > b) = 0.05$

A).

Given that,

$$f(x) = 3x^2,\ 0 \le x \le 1$$

$$P(X \le a) = P(X > a)$$

$$P(X > b) = 0.05.$$

$$P(X \le a) = P(X > a) \Rightarrow \int_0^a f(x)\,dx = \int_a^1 f(x)\,dx.$$

$$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$$

$$\left.\frac{3x^3}{3}\right]_0^a = \left.x^3\right]_a^1$$

$$a^3 = 1 - a^3$$

$$2a^3 - 1 = 0$$

$$2a^3 = 1$$

$$a^3 = 1/2$$

$$a = 0.7937$$

$$P(X > b) = 0.05 \Rightarrow \int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$\left.x^3\right]_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = 0.95$$

$$b = \underline{\underline{0.983}}$$

Continuous Distributions.

Uniform Random variable

* If we consider an experiment in which the outcome is constrained to lie in a known interval [a,b] and all outcomes are equally likely [All having same probability].

* The probability density fn of such a r.v is constant over the interval [a,b] and is zero else where.

ie, $$f(x) = \begin{cases} c, & a \le x \le b \\ 0, & o.w \end{cases}$$

By normalization condition,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{a}^{b} c\,dx = 1$$

$$c[b-a] = 1$$

$$c = \frac{1}{b-a}$$

Def:

A continuous r.v X with pdf,

$$f(x) = \begin{cases} 1/b-a, & a \le x \le b \\ 0 & o.w \end{cases}$$

is called a uniform r.v. We also say that X is uniformly distributed on [a,b] and we denote this by writing $X \sim \text{Uniform}(a,b)$ or briefly $X \sim U[a,b]$

Mean & variance of uniform R.V

pdf of uniform rv, $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o.w \end{cases}$

$$E(X) = \int_a^b \frac{1}{b-a} \cdot x \, dx$$

$$= \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \frac{b^2-a^2}{2}$$

$$E(X) = \frac{b+a}{2}$$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$= \frac{1}{b-a} \left[\frac{b^3-a^3}{3} \right]$$

$$= \frac{(b^2-a^2)(b^2-ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$\text{Var}(x) = E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4(b^2+ab+a^2)}{12} - \frac{3(b^2+2ab+a^2)}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12}$$

$$V(x) = \frac{(b-a)^2}{12}$$

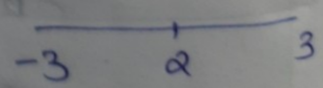
For a r.v $x \sim U(a,b)$

Mean $E(x) = \frac{b+a}{2}$

Variance $= \frac{(b-a)^2}{12}$

2. If r.v x has a uniform distribution in $(-3,3)$. Find $P(|x-2|<2)$

A). $$\text{pdf} = \begin{cases} 1/b-a, & a \le x \le b \\ 0, & \text{o.w} \end{cases}$$



$$\text{pdf} = \begin{cases} 1/6, & -3 \le x \le 3 \\ 0, & \text{o.w} \end{cases}$$

$P(|x-2|<2)$

$|x-2| = -2 < x-2 < 2$

$-2+2 < x < 2+2$

$0 < x < 4$

$P(0<x<4) = \int_0^3 f(x)\,dx + 0$

$= \int_0^3 \frac{1}{6}\,dx$

$= \frac{1}{6} x \Big]_0^3$

$= \frac{1}{6}[3-0]$

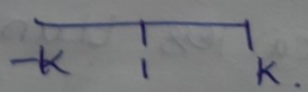
$= \frac{1}{2}$ //

2. An rv x uniformly distributed on the interval $(-k,k)$. Find k if $P(x \geq 1) = \frac{1}{3}$

A. $x \sim u(-k,k)$

k?

$P(x \geq 1) = \frac{1}{3}$



$P(x \geq 1) = \int_1^K f(x)\,dx$

$f(x) = \begin{cases} \frac{1}{b-a}, & -k \leq x \leq k \\ 0, & o.w \end{cases}$

$$f(x) = \begin{cases} \frac{1}{2K}, & -K \le x \le K \\ 0, & \text{o.w} \end{cases}$$

$P(X \geq 1) = \frac{1}{3}$

$$\cancel{\int_1^K \frac{1}{2K}\,dx = \frac{1}{3}}$$

$$\cancel{\frac{1}{3}\int_1^K dx = \frac{1}{3}}$$

$$\cancel{K-1 = 1}$$

$$\cancel{K = 2}$$

$$\int_1^K \frac{1}{2K}\,dx = \frac{1}{3}$$

$$\frac{1}{2K}\int_1^K dx = \frac{1}{3}$$

$$\frac{1}{2K}[K-1] = \frac{1}{3}.$$

$$\frac{K-1}{2K} = \frac{1}{3}$$

$$3(K-1) = 2K.$$

$$3K - 3 = 2K.$$

$$K = 3$$

2. Buses arrive at a specified stop at 15 min intervels, starting at 7am. If a passenger arrives at the stop at a random time that is uniformly distributed b/w 7am and 7.30am. Find the probabilities that he waits

(a) less than 5 minutes for a bus

b) atleast 10 minutes for a bus.

A). Let x denote the length of time (in minutes) b/w 7am & the passengers arrival time.

Then $X \sim U[0, 30]$

pdf of $X = \begin{cases} 1/30, & 0 \le x \le 30 \\ 0, & ow \end{cases}$

a). we have to find the probability that the passenger waits less than 5 minutes for a Bus.

ie,

$P(X<5) = ?$

The bus arrives at specified stop at 7am, 7.15 am & 7.30 am.

∴ the passenger wait for a bus, less than 5 minutes means, either

~~he arrives at stop atleast at 7.10 am~~

he arrives at stop b/w 7.10 AM and 7.15 AM

or b/w 7.25 am and 7.30 am.

ie,

$$P(X<5) = \int_{10}^{15} f(x)\,dx + \int_{25}^{30} f(x)\,dx$$

$$= \int_{10}^{15} 1/30\,dx + \int_{25}^{30} 1/30\,dx$$

$$= 1/30 \times 5 + 1/30\,[5]$$

$$= 1/6 + 1/6 = 2/6 = 1/3$$

b) here, we have to find the probability that he waits atleast 10 minutes for a bus.

∴ he arrives at stop either in b/w 7.0am and 7.15am or in b/w 7.15 am & 7.20 am

we've to find

$\int_{5}$ les^ $P(X \geq 10)$

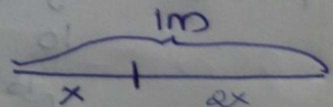
$$P(X \geq 10) = \int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx$$

$$= \frac{1}{30} \times 5 + \frac{1}{30} \times 5$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

2. A string 1 meter long, is cut into two pieces at a random point b/w its ends. what is the probability that the length of one piece is atleast twice the length of the other?



A). Suppose that the string is placed along the x axis such that its ends are at $x=0$ & $x=1$. Let x is the distance to the point where string is cut. Then x follow uniform distribution in the interval [0,1] with pdf

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & o.w \end{cases}$$

we've to find the probability that the length of one piece is atleast twice the length of other.

let, length of one piece $= x$

" " other piece $= 1-x$

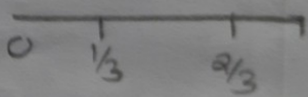
$\therefore$ we've to find the probability that,

$$P(1-x \geq 2x) = P(1 \geq 3x)$$
$$= P(3x \le 1)$$
$$= P(x \le 1/3).$$

~~here we can cut the string at 2~~.

here we can able to cut the string at both ends. $\therefore$ The probability is given by,

$P(x \le 1/3)$ or $P(x \geq 2/3)$.



ie, $P(x \le 1/3) + P(x \geq 2/3)$

$$\int_0^{1/3} 1\, dx + \int_{2/3}^{1} 1\, dx$$

$$= 1/3 + [1 - 2/3] = \underline{\underline{2/3}}$$

Q. X is uniformly distributed with mean 1 and variance 4/3. If 3 independent observations of X are made, what is the probability that all of them are negative.

A) Given that,

mean = 1 $\left(\frac{b+a}{2}\right)$

variance = 4/3 $\left(\frac{(b-a)^2}{12}\right)$

$\frac{b+a}{2} = 1 \Rightarrow b+a = 2$

$\frac{(b-a)^2}{12} = 4/3$

$(b-a)^2 = 16$

$b-a = \pm 4$

$b+a = 2$
$b-a = \pm 4$ $\Rightarrow$ $a = 2-b$

$b-(2-b) = \pm 4$

$b-2+b = \pm 4$

$2b = \pm 4+2$

$b = 3$ or -1

$a = -1$ & $b = 3$ or $a = 3$ & $b = -1$

$X \sim U(-1, 3)$

$$f(x) = \begin{cases} 1/4 & -1 \le x \le 3 \\ 0 & , \text{ o.w.} \end{cases}$$

probability of one observation to be -ve =

$$P(X<0) = \int_{-1}^{0} f(x)\,dx$$

$$= \frac{1}{4}[0+1]$$

$$= \frac{1}{4}$$

If three independent observations are made, probability that all of them -ve

$$= P(X<0) \times P(X<0) \times P(X<0)$$

$$= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$$

* Two events are independent iff $P(A \cap B) = P(A) \cdot P(B)$

? A passenger arrives at a bus stop at 10.00 AM knowing that the bus will arrive at a random time b/w 10.00 AM and 10.30 A.M. What is the probability that he will have to wait longer than 10 minutes ?

A). Let x denote the length of time b/w 10.00 Am and 10:30 Am and the ~~passenger~~ bus arrival time, i.e.

$$X \sim U[0, 30]$$

we've to find the probability that the passenger will have to wait longer than 10 minutes.

given that, the passenger arrives at bus stop at 10:00 AM.

∴ we've to find,

$P(X>10) = ?$

$$P(X>10) = \int_{10}^{30} f(x)\, dx.$$

here, X follows uniform distribution,

pdf of uniform distribution $= \frac{1}{b-a}$ $x \geq 0$

$$f(x) = \frac{1}{30}$$

$$P(X>10) = \int_{10}^{30} \frac{1}{30}\, dx$$

$$= \frac{1}{30}\left[30-10\right]$$

$$= 2/3$$

Exponential Random Variable:

A exponential r.v is usually used to model time b/w occurrences of certain types of events in an interval of time, the events must be happening independent of one another over a time interval at constant rate.

* Some examples of events that can be modelled by an exponential distribution are,

(1). Time b/w successive failures of a machine.

(2) Time b/w " arrivals of customers in a bank.

(3) Time elapsed b/w successive occurrences of earth quake in a region.

Definition

A continuous r.v X with probability density fn

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o.w \end{cases}$$

where $\lambda > 0$, is called an exponential r.v Also we say that x follows an exponential distribution with parameter λ and we write $x \sim exp(\lambda)$

Note:

If $X \sim exp(\lambda)$

for $x > 0$ $f(x) = \lambda e^{-\lambda x}$

$$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{x} e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{-e^{-\lambda x}}{\lambda}\right]_{0}^{x}$$

$$= -e^{-\lambda x} - \left[-e^{-\lambda \cdot 0}\right]$$

$$= -e^{-\lambda x} + 1$$

$$= \underline{\underline{1 - e^{-\lambda x}}}$$

$$P(X > x) = 1 - P(X \le x)$$

$$= 1 - (1 - e^{-\lambda x})$$

$$= 1 - 1 + e^{-\lambda x}$$

$$= \underline{\underline{e^{-\lambda x}}}$$

$$P(X \le x) = 1 - e^{-\lambda x}$$

$$P(X > x) = e^{-\lambda x}$$

Mean and variance

Let $X \sim Exp(\lambda)$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \cancel{x \cdot \frac{-\lambda e^{-\lambda x}}{\lambda}}\ \lambda e^{-}$$

$$= \lambda \int_0^\infty x\, e^{-\lambda x}\, dx$$

$$= \lambda \left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= -x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \Big]_0^\infty$$

$$= \underline{\underline{1/\lambda}}$$

$E(x) = 1/\lambda$

$$E(x^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{x^2 e^{-\lambda x}}{-\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} + \frac{2 e^{-\lambda x}}{-\lambda^3} \right]_0^\infty$$

$$= -x^2 e^{-\lambda x} - \frac{2x e^{-\lambda x}}{\lambda} - \frac{2 e^{-\lambda x}}{\lambda^2} \Big]_0^\infty$$

$$= 2/\lambda^2$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= 2/\lambda^2 - 1/\lambda^2$$

$$= 1/\lambda^2$$

x is an exponential r.v with parameter λ,

$E(X) = 1/\lambda$ and $V(X) = 1/\lambda^2$.

2. The life time (in years) of an electronic component is an exponential variable with mean 1 year. Find the lifetime 'L' which a typical component is 60% certain to exceed

A). Let $x \sim \exp(\lambda)$ denotes the lifetime of the component with mean $x = 1$.

$\lambda = 1$

$p(x > L) = 60\% = 0.6$

$p(x > L) = \int_L^\infty f(x)dx = 0.6$

$= \cancel{\int_L^\infty \lambda e^{-\lambda x} = 0.6}$

$= \cancel{\lambda}\ e^{-\lambda L} = 0.6$

$-L\lambda = \log(0.6)$

$-L = \log(0.6)$

$L = -\log(0.6)$

$= 0.51$

2. If X follows exponential distribution with $p(X\le 1) = p(X>1)$, Find mean & variance.

A) $p(X\le 1) = p(X>1) \Rightarrow 1-e^{-\lambda} = e^{-\lambda}$

$1 = 2e^{-\lambda}$

$e^{-\lambda} = 1/2.$

$-\lambda = \log(1/2)$

$= \log 1 - \log 2.$

$\lambda = \log 2$

mean $= 1/\log 2$

variance $= 1/2\log 2$

?. If the distance that a car can run before its battery wears out is exponentially distributed with an avg value of 5000 km. If the owner desires to take a 2000 km trip, what is the probability that he will be able to complete his trip without having to replace the car battery.

A). Let $X \sim \exp(\lambda)$ denote the distance covered by the car before the battery wears out.

given that

$E(X) = 1/\lambda = 5000$ km

$\lambda = 1/5000$

$P(X > 2000) = ?$

$= \int_{2000}^{\infty} f(x)\,dx$

$= e^{-\lambda x}$

$= e^{-1/5000 \times 200}$

$= e^{-2/5}$

11/4/

1. Commonly, car cooling systems are controlled by electrically driven fans. Assuming that the lifetime T in hours of a particular make of fan can be modelled by an exponential distribution with $\lambda = 0.0003$. Find the proportion of fans which will give at least 10000 hours service. If the fan is redesigned so that its lifetime may be modelled by an exponential distribution with $\lambda = 0.00035$, would you expect more fans or less to give at least 10000 hours service?

A). Let $X \sim \exp(\lambda)$ denotes the lifetime T in hours of a particular make of fan with $\lambda = 0.0003$.

a) we've to find $P(X \geq 10000)$

$$P(X > 10000) = e^{-\lambda x}$$
$$= e^{-0.0003 \times 10000}$$
$$= e^{-3} = \underline{\underline{0.0498}}$$

hence about 5% of the fans may be expected to give at least 10000 hours service.

After the redesign,

$\lambda = 0.00035$,

$$P(X > 10000) = e^{-\lambda x} = e^{-0.00035 \times 10000}$$
$$= e^{-3.5}$$
$$= 0.0301$$

3% of the fans may be expected to give atleast 10000 hours service.

$\therefore$ we expect less fans to give atleast 10000 hours service, after redesign.

2 The Time b/w telephone calls that arrive at a switch board are exponentially distributed with mean of 30 minutes. Given that a call has just arrived, what

is the probability that it takes at least 2 hours before the next call arrives.

A). Let $X \sim \exp(\lambda)$, denotes the time b/w the telephone calls that arrives at a switch board.

mean $= 1/\lambda = 30$ $\qquad \frac{30}{60}$ $\frac{1}{2}$ hr.

$\lambda = 1/30$ $\qquad \lambda = 2$

we have to find. $P(X > 120)$ $\quad P(X>2) = e^{-2\times 2}$

$P(X>120) = e^{-\lambda x}$ $\qquad = e^{-4}$

$= e^{-1/30 \times 120}$

$= e^{-4} = \underline{\underline{0.0183}}$

2. The PDF of the T in weeks b/w thunderstorms in a certain place is given by.

$f(t) = 0.02 e^{-0.02t}, \; t \geq 0$

a) what is the expected time b/w thunderstorms?

b) find $P(T \leq t \mid T < 40) \; \forall t$

c) Find $P(40 < t < 60)$

A) Given, $f(t) = 0.02 e^{-0.02t}, \; t \geq 0.$

Let $T \sim \exp(0.02)$, $t \geq 0$

$\lambda = 0.02$

a) we've to find $E(T) = ?$

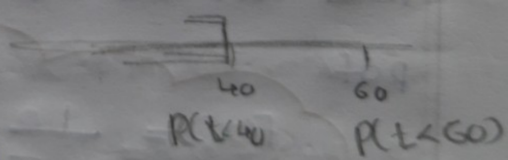
$E(T) = 1/\lambda = 1/0.02 = \underline{\underline{50 \text{ weeks}}}$

expected time b/w thunderstorms = 50 weeks.

b) $P(T \leq t \mid T < 40) = ?$

$P\left(\frac{\cancel{T \leq t}}{T<40}\right)$.

c) $P(40 < t < 60) =$



$$P(t<60) - P(t<40)$$

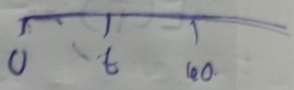
$$= 1 - e^{-\lambda x} - [1 - e^{-\lambda x}]$$

$$= 1 - e^{-0.02 \times 60} - 1 + e^{-0.02 \times 40}$$

$$= -e^{-1.2} + e^{-0.8}$$

$$= \underline{\underline{0.1481}}$$

2) $P(B \backslash A) = \dfrac{P(A \text{ and } B)}{P(A)}$ — conditional probability



$P(T \le t \backslash T < 40) = P($

NORMAL DISTRIBUTION.

Normal (Gaussian) r.v

* The most imp r.v is the normal r.v or
* Normal distribution.
* It is a continuous r v used to model data which tend to concentrate around the mean value.

For eg: The marks scored by 1000 students in a competitive examination, which has a tendency to accumulate near a mean value.

Defn:

The normal r.v X is a continuous r.v with pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \; \sigma > 0$$

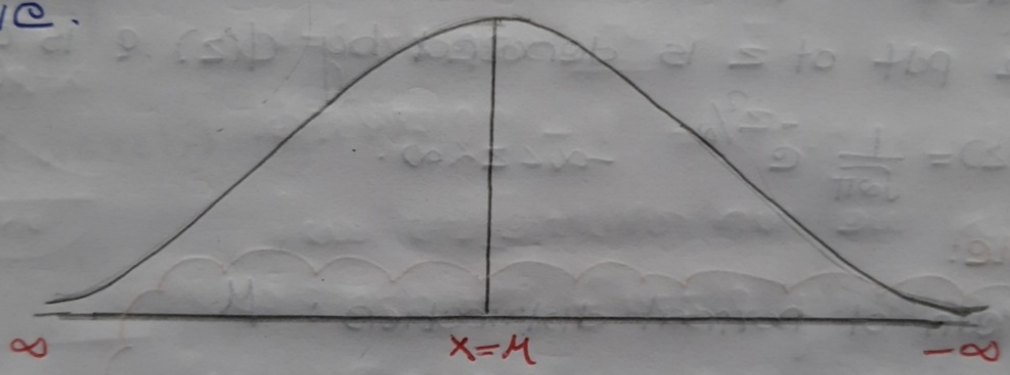
we also say that x has a normal distribution. The constants μ and σ are the parameters of the distribution.

* where x follow normal distribution we write it symbolically as

$x \rightarrow N(\mu, \sigma)$ or $x \rightarrow N(\mu, \sigma^2)$

properties of Normal distribution.

The graph of the normal distribution given by, is a bell shaped smooth symmetrical curve known as the normal curve.



1. The normal curve is symmetric about the co-ordinate at $x=\mu$, ie, $f(\mu+c) = f(\mu-c)\ \forall c$.
2. The normal curve $f(x)$ has a maximum at $x=\mu$ and the maximum value of the co-ordinate is $\frac{1}{\sigma\sqrt{2\pi}}$
3. The mean, median & mode are identical.
4. The normal curve extends from $-\infty$ to $+\infty$
5. The curve touches x axis only at $\pm\infty$

Std Normal distribution.

When $X \to N(\mu, \sigma)$, its pdf is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Define $z = \frac{X-\mu}{\sigma}$, By changing the variable,

with $\mu = 0$ & $\sigma = 1$

is called the std normal or normalized Gaussian

~~$f(z) =$ fc~~

~~and is~~

The pdf of z is denoted by $\phi(z)$ & is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty.$$

Note:

Mean of normal distribution : μ

Variance " " " : σ^2

Computation of probabilities

If $X \sim N(\mu, \sigma^2)$, then.

$$P(a \le X \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$z = \frac{x-\mu}{\sigma}$

$dz = \frac{1}{\sigma} dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-z^2/2} \, \sigma \, dz$$

when $x = a$, $z = \frac{a-\mu}{\sigma}$

$x = b$, $z = \frac{b-\mu}{\sigma}$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \int_{z_1}^{z_2} \phi(z) \, dz$$

$$= p\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma} \right)$$

Result:

If $X \sim N(\mu, \sigma^2)$, we can convert probabilities involving x into probabilities involving std normal variable z using,

$$p(a \le x \le b) = p\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma} \right)$$

* we can evaluate the std normal probabilities with the help of normal table.
* Typically a std normal table gives the area under the std normal curve $\phi(z)$, correct up to four decimal places, for various values of z in the range $0 \le z \le 4$ in small steps 0.01

ie, Table gives us the probabilities of the form $p(0 \le z \le a)$ for $a \le 4$.

* Probabilities for z outside this range can be computed by converting into the form $p(0 \le z \le a)$ using following properties.

1. The curve $y = \phi(z)$ is symmetric w.r.t y-axis. Hence the area under the curve b/w $z=0$ & $z=a$ is same as the area b/w $z=0$ & $z=a$

i.e, $P(0 \le z \le a) = P(-a \le z < 0)$

2. $P(z \geq 0) = 0.5$ & $P(z \leq 0) = 0.5$

$\{$ since $P(-\infty < z < \infty) = 1 \Rightarrow$

$2P(z \geq 0) = 1$

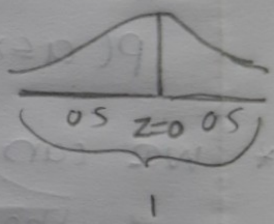
$P(z \geq 0) = 1/2$

$p(z=0) =$

$z = 0 \Rightarrow \frac{x-\mu}{\sigma} = 0$

$\Rightarrow x = \mu$

$x = \mu \Rightarrow z = 0$



Thus we can calculate the tail probability $P(z > a)$ for $a > 0$ as

$P(z > a) = 0.5 - P(0 \le z < a)$

3. $P(z \leq -4)$ & $P(z \geq 4)$ are practically zero (i.e, why tables do not give probabilities outside this range)

2. Let x be a normal r.v with mean $= -3$ & variance 4: Find

(1) $P(1 \le x < 2)$

2) $P(x > -1.5)$

3) $P(x < -3)$

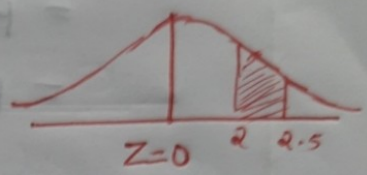
4) $P(|x+3| < 2)$

5) $P(|x+5| > 1)$

A) Let $X \sim N(\mu, \sigma^2)$ with $\mu = -3$, $\sigma^2 = 4$.

X can be converted to std normal z by

$$Z = \frac{X - \mu}{\sigma} = \frac{X+3}{2}$$

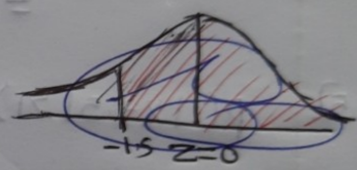


(1) $P(1 \le X \le 2) = P(2 \le Z \le 2.5)$

$= P(0 \le Z \le 2.5) - P(0 \le Z < 2)$

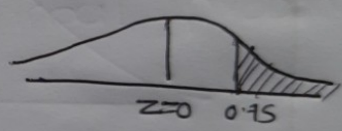
$= 0.4938 - 0.4772$

$= \underline{0.0166}$



2) $P(X > -1.5) = P\left(Z > \frac{-1.5+3}{2}\right)$

$= P(Z > 0.75)$



$= 0.5 - P(0 < Z < 0.75)$

$= 0.5 - 0.2734$

$= 0.2266$

3) $P(X < -3) = P\left(Z < \frac{-3+3}{2}\right)$

$= P(Z < 0)$

$= 0.5$

4) $P(|X+3| < 2) = P(-2 < X+3 < 2)$

$= \cancel{P\left(\frac{-2+3}{2} < \frac{X+3+3}{2} < \frac{2+3}{2}\right)}$

$= \cancel{P(-5/2 < Z < -1/2)}$

$= \cancel{P(-2.5 < Z < -0.5)}$

$= P(-2 < x+3 < 2)$

$= P(-2-3 < x < 2-3)$

$= P(-5 < x < -1)$

$= P\left(\frac{-5+3}{2} < z < \frac{-1+3}{2}\right)$

$= P(-1 < z < 1)$

$= 2 \times P(0 < z < 1)$

$= 2 \times 0.3413$

$= \underline{\underline{0.6826}}$

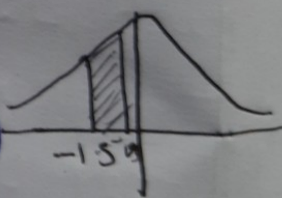
5). $P(|x+5| > 1) = 1 - P(|x+5| \leq 1)$

$= \cancel{1 - [P(-1<}$

$= 1 - P(-1 < x+5 < 1)$

$= 1 - P(-6 < x < -4)$

$= 1 - P\left(\frac{-6+3}{2} < z < \frac{-4+3}{2}\right)$



$= 1 - P(-1.5 < z < -0.5)$

$= 1 - [P(0<z<1.5) - P(0<z<0.5)]$

$= 1 - [0.4332 - 0.1915]$

$= \underline{\underline{0.7583}}$

? suppose that the life time of bulbs produced by a company are normally distributed with mean 1000 hours and std deviation 100 hours. Is this company correct when it claims that 95% of its bulbs last atleast 900 hours?

A). Let X denote the lifetime of a bulb.

The company claims that $P(X \geq 900) = 95\%$

Here $X \sim N(\mu, \sigma^2)$ with $\mu = 1000$ & $\sigma = 100$

~~$P(X \geq 900)$~~ -

we can change to std normal distribution

by $z = \frac{X-\mu}{\sigma} = \frac{X-1000}{100} \sim N(0,1)$

$P(X \geq 900) = P\left(\frac{X-1000}{100} \geq \frac{900-1000}{100}\right)$

$= P(Z \geq -1) = 0.8413$

Then 84.13 % of the bulbs would last more than 900 hours & the companys claim is false.

2). If X is a normal random variable with mean 50 and std deviation 10. Find α and β such that $P(X < \alpha) = 0.1$ & $P(X > \beta) = 0.05$

A) suppose that $X \sim N(\mu, \sigma^2)$ where $\mu = 50, \sigma = 10$

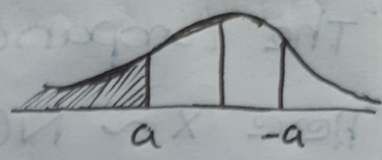
we can change to std normal variable using

$z = \frac{X-\mu}{\sigma} = \frac{X-50}{10}$ then $z \sim N(0,1)$

$P(X < \alpha) = 0.1 \Rightarrow P\left(z < \frac{\alpha - 50}{10}\right) = 0.1$

$\Rightarrow P(z < a) = 0.1$, where , $a = \frac{\alpha - 50}{10}$

Since $p(z<a) = 0.1 < 0.5$, a ~~less~~ lies to the left side of the origin.



$\therefore p(z<a) + p(a<z<0) = 0.5$

$0.1 + p(0<z<-a) = 0.5$

$p(0<z<-a) = 0.4$

From table, $-a = 1.28$ → 0.4 ന് നേരെ കിടക്കുന്ന value equal difference ആണെങ്കിൽ, take bigger one

$a = 1.28$

$\frac{\alpha - 50}{10} = 1.28$

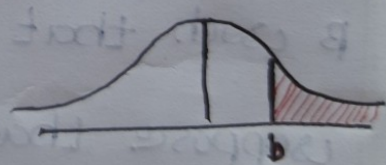
$\alpha - 50 = 12.8$

$\alpha = 37.2$

$P(X>\beta) = 0.05 \Rightarrow P\left(z > \frac{\beta-50}{10}\right) = 0.05$

$\Rightarrow P(z > b) = 0.05$, where $b = \frac{\beta - 50}{10}$

which shows that b lies right side of the origin



$P(0<z<b) = 0.5 - P(z>b)$

$= 0.45$

$\therefore$ From table, $b = 1.65$

ie, $\frac{\beta - 50}{10} = 1.65$

$\beta - 50 = 16.5$

$\beta = 66.5$

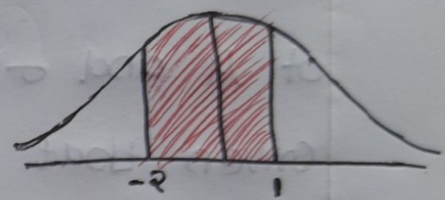
3. If X is normally distributed with mean 1 and variable 4

(1) find $p(-3<X<3)$ and

(2) obtain K if $p(X\leq K)=0.9$

A) $X\sim N(\mu,\sigma^2)$ where $\mu=1$ & $\sigma=2$

$\therefore Z=\frac{X-1}{2}\sim N(0,1)$



$p(-3<X<3)=p\left(-\frac{3-1}{2}<Z<\frac{3-1}{2}\right)$

$=p(-2<Z<1)$

$=p(0<Z<1)+p(0<Z<2)$

$=0.3413+0.4772$

$=\underline{\underline{0.8185}}$

(b) $p(X\leq K)=0.9 \Rightarrow p\left(Z\leq\frac{K-1}{2}\right)=0.9$

$\Rightarrow p(Z\leq K_1)=0.9$, where $K_1=\frac{K-1}{2}$

$p(0<Z<K_1)=0.9-0.5$

$=0.4$

$\frac{K-1}{2}=1.29$

$K=\underline{\underline{3.58}}$

4) The marks obtained by a batch of students in a certain subject are normally distributed. 10% of students got less than

45 marks while 5% got more than 75. find the percentage of students with score b/w 45 & 60.

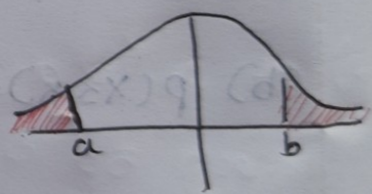
A). Let x denote the marks obtained by a student, and assume that x has mean μ and variance σ^2. we first find the radius of μ and σ.

Given that $p(x<45) = 0.1$ & $p(x>75) = 0.05$

ie, $p\left(z < \frac{45-\mu}{\sigma}\right) = 0.1$ $\quad p\left(z > \frac{75-\mu}{\sigma}\right) = 0.05$.

Let $a = \frac{45-\mu}{\sigma}$ $\quad b = \frac{75-\mu}{\sigma}$

$\therefore p(z<a) = 0.1$ $\quad p(x>b) = 0.05$



since $p(z<a) = 0.1 < 0.5$

$a<0$

$p(z>b) = 0.05 < 0.5 \therefore b>0$

$p(a<z<0) = 0.4$

$\Rightarrow p(0<z<-a) = 0.4$

$\Rightarrow -a = 1.28$

$a = -1.28$

$\frac{45-\mu}{\sigma} = -1.28$ ——— (2)

From figure, $p(a<z<b) = 0.45$

$\Rightarrow b = 1.65$

$\Rightarrow \frac{75-\mu}{\sigma} = 1.65$ —— (2)

from (1) & (2)

$$\frac{\frac{45-\mu}{\sigma}}{\frac{75-\mu}{\sigma}} = \frac{-1.28}{1.65}$$

$$\frac{45-\mu}{75-\mu} = -0.7758$$

$$45-\mu = -58.185 + 0.7758\mu$$

$$45 + 58.185 = 1.7758\mu$$

$$\mu = 58.1062$$

$$\frac{45-\mu}{\sigma} = -1.28$$

$$\frac{45-58.1062}{\sigma} = -1.28$$

$$\sigma = 10.24$$

$$P(45 < x < 60) = P(-1.28 < z < \underset{\sim 0.19}{0.1855})$$

$$= 0.3997 + 0.07535$$

$$= \underline{\underline{0.475}}$$

$\therefore$ 47.5% students score marks be 45 & 60